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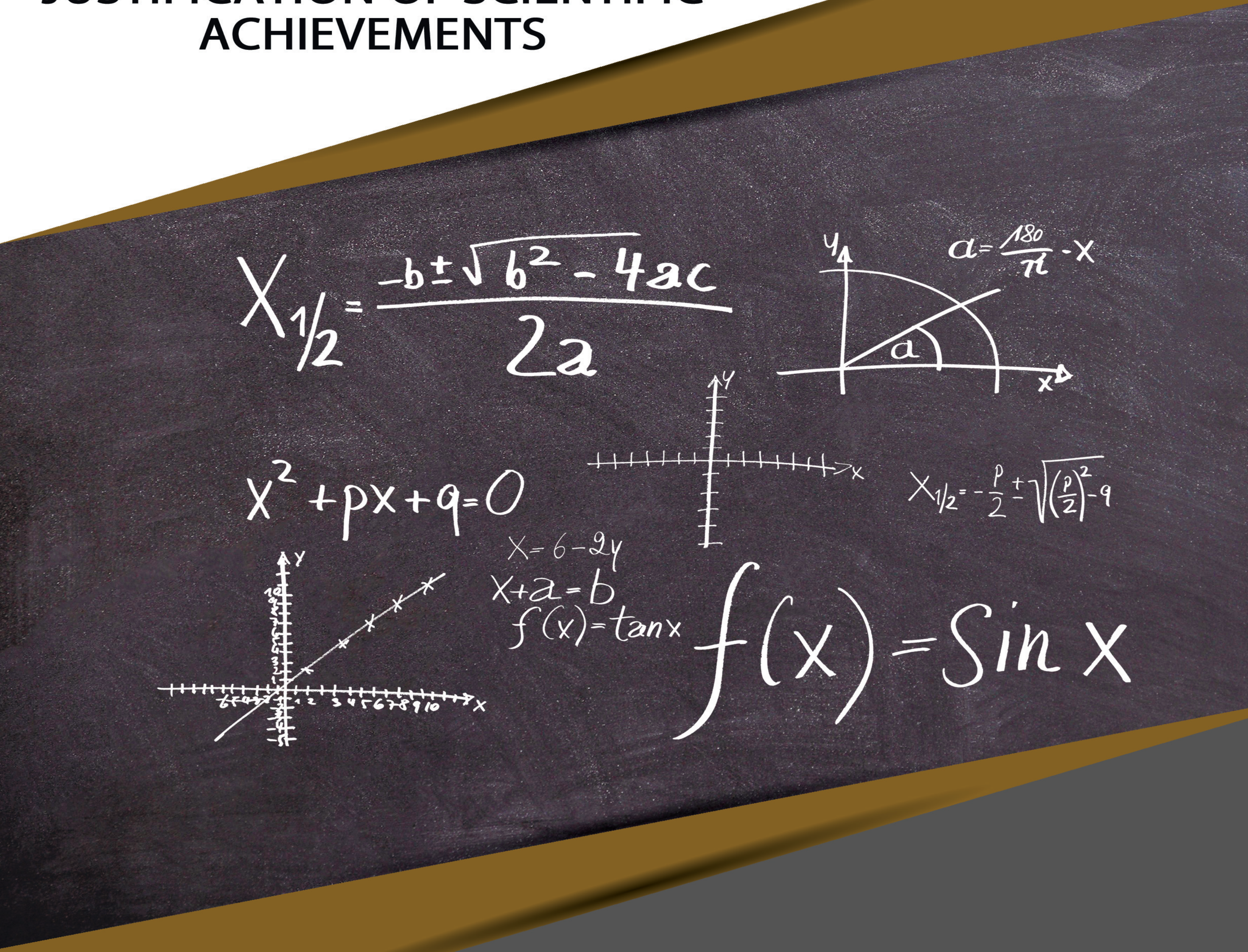

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Authors - Волосова Н.М., Мірошник Н.П., Tashchuk V., Malinevska-Biliichuk O., Ivanchuk P., Tashchuk M., Ivanchuk M., Krykun H. Ivan, Kulishov S., Pistun Y., Matiko H., Krykh H., Matiko F., Безрук В.М., Жиленко Т.І., Ярецька Н.О., Hevko R., Trokhaniak A., Zalutskyi S., Stanko A., Філіпович Ю.Ю.

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TABLE OF CONTENTS

3.2 The arcsine laws in the modelling of the natural processes depending on random factors

1.1. Random processes. Wiener process.

The process is called a phenomenon that develops, changes over time. A random process is a process whose development depends, among other things, on random factors. Random processes are models, mathematical forms of description, and study of the behaviour of many natural phenomena, such as physical, biological, socio-economic ones. Examples of random processes include deviations in the processing of parts from standard sizes, the service life of household appliances and technical equipment, fluctuations in exchange rates and securities prices, the number of orders for service over time in queuing systems, and so on.

Such a wide practical application of random processes, of course, led to their active study and practical use of the results. However, their study is not easy.

One of the first random processes considered and studied by scientists was the microscopic observation of the behaviour of solid particles in liquids or gases the process of chaotic thermal motion of these particles under the action of uneven impacts of molecules of matter from different directions, which was called Brownian[1] motion.

Brownian motion has an unexpected feature for scientists of that time - the motion of particles never stops. So the reason for the motion of particles is precisely the structure of a liquid or gas. Experiments have shown that the velocity of particles during Brownian motion increases with increasing temperature of the liquid or gas, which suggests the connection of Brownian motion with the chaotic thermal motion of the particles that make up the liquid or gas. The discovery and explanation of Brownian motion were of great importance for physics, as it became both experimental and theoretical confirmation of the chaotic thermal motion of molecules and atoms, which is a fundamental position of the modern molecular kinetic theory of the structure of matter.

[1] It is named in honour of the Scottish biologist Robert Brown, who in 1827, when examined under a microscope, noticed the chaotic and incessant movement of plant pollen particles in water.

The mathematical model of the Brownian motion was created at the beginning of the twentieth century almost simultaneously by M. Smoluchowski in 1904 in Lviv (Ukraine) and A. Einstein in 1905 in Bern (Switzerland). These scientists independently developed theories that described the parameters of the motion of molecular particles in matter.

The American mathematician N. Wiener made a significant contribution to the further study of the Brownian motion. It was in his honour that the Wiener process was named – it is denoted by $w(t)$ – a mathematical model of the Brownian motion process or, in general, of random walk with continuous time.

The Wiener process $w(t)$ in the theory of random processes is one of the most important random processes and plays an important role in applied mathematics. A detailed review of both the Wiener process and its properties and its application in socio-economic processes can be found in [50].

1.2. Stochastic differential equations.

The Wiener process, as a mathematical model of Brownian motion, proved to be an extremely complex mathematical object. One of the key and fundamental properties of the Wiener process $\boldsymbol{w}(\boldsymbol{t})$ was that, firstly, it is not differentiated (it does not have a normal differential) at any point, and, secondly, has an unlimited variation on any which segment. This makes it impossible to study the Wiener process using conventional tools (differential, derivative, integral) of mathematical analysis.

However, in the 1940s the Japanese mathematician Kiyosi Itô and, independently of him, the Ukrainian mathematician Iosif Gikhman, developed the concept of a stochastic differential of a random process, denoted by $dw(t)$ for the Wiener process (see, for example, [51]), which is analogous to the concept of the ordinary differential in mathematical analysis.

This not only made it possible to construct a theory of stochastic Ito calculus (i.e., the ability to find differentials and integrals from random processes) but also made it possible to consider random processes as solutions of differential equations of a certain type – which includes stochastic differentials in addition to ordinary differentials. Such differential equations became known as stochastic differential equations.

For example, let us consider $\xi(t)$ – a random continuous-time process with continuous trajectories. One can prove (see, for example, [52]) that process $\xi(t)$ is a solution of the following stochastic differential equation:

$$\xi(t) = \xi(0) + \int_0^t b(\xi(s))ds + \int_0^t \sigma(\xi(s))dw(s)\ , \tag{1.1}$$

where $\int_0^t b(\xi(s))ds$ be an ordinary Lebesgue integral and $\int_0^t \sigma(\xi(s))dw(s)$ be an Ito stochastic integral over a stochastic differential $dw(s)$. Moreover, the coefficient $b(x)$ for an ordinary differential is called a drift coefficient, and the coefficient $\sigma(x)$ for a stochastic differential is called a diffusion coefficient.

Subsequently, many results were obtained regarding the conditions of existence and unity of solutions of stochastic differential equations of the form (1.1), their properties, and methods of solving certain types of stochastic differential equations.

A detailed review of the theory of stochastic differential equations developed research methods and the results obtained can be found in [52], [53], [54].

1.3. Random processes with the local time at some point

Here is a brief overview of the issue, using [54], [55].

With the progress of science and technology, the development of their applications such as filtering, encrypting, and decrypting signals in communication networks, forecasting various economic and social phenomena and indicators, calculations involving a large number of insignificant factors, etc. are becoming increasingly popular. Modern physics in the study of many phenomena requires taking into account various kinds of fluctuation effects. These can be thermal noise, instability, turbulence, inhomogeneity of the environment, problems in statistical hydrodynamics, statistical radio physics and acoustics, and so on.

If the phenomenon under consideration occurs in a multilayer medium, such as liquid filtration, then the so-called local time of process appears in the mathematical model of the phenomenon. The local time of random process at a certain point to some instant is a purely probabilistic object. The study of random processes with local time at a certain point is a modern area of research in the theory of random processes.

It has been proved that the local time of a non-random process at a certain point for any period of time is identically equal to 0, but for random processes, the local time exists and it is different from 0. For example, for a random process $\xi(t)$ given as a solution of stochastic differential equation (1.1) local time at point a until time t is defined as follows:

$$L^{\xi}(t,a) = \lim_{\delta\to 0}\frac{1}{2\delta}\int_0^t I_{\{(a-\delta;\, a+\delta)\}}\big(\xi(s)\big)\sigma^2(\xi(s))\,ds\,, \tag{1.2}$$

where $I_{\{A\}}(x) = \begin{cases} 1\,, & if \quad x\in A \\ 0\,, & if \quad x\notin A \end{cases}$ be the indicator function of set A.

The first representative of this class of random processes is so-called "skew Brownian motion", which was defined by K. Itô and H. McKean and firstly productive studied by W. Rosenkrantz and M. Portenko. Summarizing the results of these and other authors, we can say that the skew Brownian motion can be considered as a solution of a stochastic equation with local time at some point a of the form

$$\xi(t) = \beta L^{\xi}(t,a) + w(t), \tag{1.3}$$

where $w(t)$ is a standard Wiener process. The solution of Equation (1.3) is interpreted by researchers as Brownian motion in some medium (liquid or gas), where at a certain level there is a partially permeable membrane.

The coefficient β at local time in Equation (1.3) is interpreted as an indicator of membrane permeability. We will talk about the reasons for this interpretation later.

2.1. Arcsine laws for the Wiener process

Let $I_{\{A\}}(x)$ be the indicator function of set A, as defined above.

Theorem A. (by French mathematician P. Levy, 1939)

Let $\{w(t),\ \ t\in[0,50]\}$ be a standard Wiener process. Then for $0\ \le\ x\ \le 1$

$$P\left\{\int_0^1 I_{\{(0;\,+\infty)\}}(w(t))dt \le x\right\} = \frac{2}{\pi}\arcsin x. \tag{2.1}$$

This result was later summarized as

$$P\left\{\frac{1}{T}\int_0^T I_{\{(0;\,+\infty)\}}(w(t))dt \le x\right\} = \frac{2}{\pi}\arcsin x. \tag{2.1.1}$$

This result is called “the arcsine law”. In simple words, we can say that according to the arcsine law, the time that the trajectory of the Wiener process spends in the upper half-plane has the arcsine distribution. That is, if the trajectory of the Wiener process has been in the lower half-plane for a long time, then the probability increases that the trajectory of the Wiener process will pass into the upper half-plane.

This property of the Wiener process and random deviations of exchange rates, stocks, securities, which have a behaviour similar to the behaviour of the Wiener process, have led to numerous attempts to use the arcsine law for prediction of the exchange rates of economic assets.

To do it strictly mathematically, of course, is impossible due to the presence of randomness. However, investors often are satisfied with approximate results, so, the number of publications that try to use the arcsine law for prediction of the behaviour of exchange rates of economic assets increase: Google returns almost 1 million results for keywords “arcsine law economics”.

Later were received other similar results where appear arcsine function. They were called the arcsine laws too; particularly the result of the Theorem A was named “the first arc-sine law”. Consequences of these works are the following results:

Theorem B. (so called “the second arc-sine law for Brownian motion”).

Let $L = sup\{t \leq 1 : w(t) = 0\}$ be the last zero of Wiener process until the instant $t = 1$. Then (for $0 \leq x \leq 1$)

$$P\{L \leq x\} = \frac{2}{\pi} \arcsin x. \tag{2.2}$$

Theorem C. (so called “the third arc-sine law for Brownian motion”).

Let $Z = arg \max_{t \in [0,1]} \{w(t)\}$ be the instant of hitting Wiener process its maximum for time $[0, 50]$. Then (for $0 \leq x \leq 1$)

$$P\{Z \leq x\} = \frac{2}{\pi} \arcsin x. \tag{2.3}$$

2.2. Arcsine laws for the skew Brownian motion

Let us consider the skew Brownian motion as a solution of a stochastic equation involving the local time of process at the point 0 until time t

$$\xi(t) = \beta L^{\xi}(t,0) + w(t), \ \ t \geq 0, \tag{2.3}$$

where $w(t)$ be a standard Wiener process.

It is well known from [56], that if $|\beta| \leq 1$ then there is the unique strong solution of Equation (2.3) i.e. a symmetric local time of process $\xi(t)$ at the point 0 until time t of the form of Equation (1.2) exists almost surely:

$$L^{\xi}(t,0) = \lim_{\delta \to 0} \frac{1}{2\delta} \int_0^t I_{\{(-\delta;\,\delta)\}}\big(\xi(s)\big)\, ds\,,$$

and Equation (2.3) fulfils almost surely.

One can prove the following results:

Theorem 2.1. (A first arc-sine law for the skew Brownian motion).

Let $\xi(t)$ be the skew Brownian motion, defined by Equation (2.3) and constant $|\beta| < 1$. Then for every $0 \leq x \leq 1$, we have

$$P\left\{\frac{1}{T}\int_0^T I_{\{(0;\,+\infty)\}}(\xi(t))dt \leq x\right\} = \frac{2}{\pi}\arcsin\sqrt{\frac{(1-\beta)^2 x}{(1+\beta)^2 - 4\beta x}}. \tag{2.4}$$

Theorem 2.2. (A second arc-sine law for the skew Brownian motion).

Let $\xi(t)$ be the skew Brownian motion, defined as above, constant $|\beta| < 1$ and

$$N_T = sup\{t \leq T :\ \xi(t) = 0\}\,.$$

Then for every $0 \leq x \leq T$, we have

$$P\{N_T \leq x\} = \frac{2}{\pi}\arcsin\sqrt{\frac{x}{T}}\ .$$

Theorem 2.3. (A third arc-sine law for the skew Brownian motion).

Let $\xi(t)$ be the skew Brownian motion, defined as above, constant $|\beta| < 1$ and

$$Z_T = arg \max_{t\in[0,T]}\{\xi(t)\}\,.$$

Then for every $0 \leq x \leq T$, we have

$$P\{Z_T \leq x\} = \frac{2}{\pi}\arcsin\sqrt{\frac{x}{T}}.$$

2.3. Consequences of the Theorem 2.1.

Corollary 1. It’s clear that if $\beta = 0$ then from Equation (2.4) we have result for Wiener process, i.e. Equation (2.1) from Theorem A. Thus, Theorem 2.1 generalizes result of Theorem A.

Corollary 2. It’s well known from [56] that if $\beta = 1$ then distributions of the skew Brownian motion coincide with distributions of process $|w(t)|$ and if $\beta = -1$ then distributions of the skew Brownian motion coincide with distributions of process $-|w(t)|$. So we should expect that for $\beta \to 1$ the probability

$$P\left\{\frac{1}{T}\int_0^T I_{\{(0;\,+\infty)\}}(\xi(t))dt \leq x\right\}$$

have to tend to 0 for $0 < x < 1$, and for β tends to -1 the probability

$$P\left\{\frac{1}{T}\int_0^T I_{\{(0;\,+\infty)\}}(\xi(t))dt \leq x\right\}$$

have to tend to 1 for $0 < x < 1$. It is obviously that exact the same results follow from Equation (2.4).

Corollary 3. It follows from Theorem A that for the Wiener process $w(t)$ holds

$$P\left\{\frac{1}{T}\int_0^T I_{\{(0;\,+\infty)\}}(w(t))dt \leq \frac{1}{2}\right\} = \frac{1}{2}.$$

Let us compare this result with similar results for the skew Brownian motion. As follows from Theorem 2.1, we have:

$$P\left\{\frac{1}{T}\int_0^T I_{\{(0;\,+\infty)\}}\big(\xi(t)\big)dt \leq \frac{1}{2}\right\} = \frac{2}{\pi}\arcsin\sqrt{\frac{(1-\beta)^2}{2+2\beta^2}}\ ;$$

$$P\left\{\frac{1}{T}\int_0^T I_{\{(0;\,+\infty)\}}(\xi(t))dt \leq \frac{(1-\beta)^2}{2+2\beta^2}\right\} = \frac{1}{2}.$$

The proof of Theorems 2.1, 2.2, 2.3, formulation and proof of some other results, and much more detailed review of publications other authors on this topic are given in the work of the author of these lines [57].

3. Some possible interpretations of the obtained results for prediction of the behaviour of natural processes depending on random factors and having a partially permeable barrier at a certain level.

Now let us recall the numerous studies of the price dynamics of some economic assets – the exchange rate, stocks, commodities, the ForEx market, etc., which contain attempts to predict these price dynamics. If the investor considers the market stable and homogeneous and assumes that fluctuations in economic assets are caused only by chance, then to predict the behaviour of exchange rates or other economic assets, the investor should (and really do) try to use the law of arcsine for the Wiener process.

Further in studies devoted to the investigation of pricing different goods, currencies, assets we often meet an idea of "psychological barrier", i.e. such price, after reaching it we observe a significant change of attitude of investors to this "goods"— an additional growth or reduction the demand on it.

On another side, it's well known ([54], p. 139) that the local time of the one-dimensional random process is like some partially permeable "membrane" such as the path of a one-dimensional random process with local time after reaching the level where the local time with the coefficient $|\beta| \leq 1$ is concentrated with probability $\frac{1+\beta}{2}$ will go up and with probability $\frac{1-\beta}{2}$ it will go down. It is these values (more precisely, one value – the constant β, i.e. the coefficient at the local time of the process) that determine the indicator of permeability of the membrane.

In extreme cases, at $\beta = \pm 1$ the membrane ceases to be permeable, becomes a barrier with reflection up (at $\beta = 1$) or down (at $\beta = -1$).

For example, suppose that about 70% of investors for *some reason* believe that the exchange rate of a certain economic asset equal to 25 $ – it's cheap. Then when the exchange rate of this asset reaches the price equal to 25 $ about 70% of investors begin to actively buy it, so the rate of this asset goes up. Approximately, we can assume that the probability of the stock price to go up in this situation (after it reaches level 25 $) is equal to 0.7, i.e. $\frac{1+\beta}{2} = 0{,}7$ so we find that $\beta = 0.4$.

That is why we assume that the skew Brownian motion can be considered as a model of the behaviour of exchange rates, stocks, or other economic assets with semipermeable membranes – it may be due to psychological, regulatory, or other reasons. Therefore, the results of this chapter can be used for predicting the exchange rate of currencies or other economic assets in the situation of the existence of some psychological, regulatory, technical, or technological partially permeable barriers.

Moreover, for the same reasons, it is natural to assume that the results of this chapter can be used to modelling and predicting the behaviour of many other natural processes, such as demographic, electoral, biological, technical ones in the situation of the presence of permeable barriers in a mathematical model of these processes.