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About convergence of solutions of one-dimensional stochastic equations

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In this paper, we consider a random process as a solution of stochastic differential equations with dependence of the coefficients on small parameter ε and we suppose that the drift coefficients of these equations are unbounded on the parameter ε. We consider more general requirements on the convergence of some functions of coefficients of stochastic equations to limit functions. Necessary and sufficient conditions for the weak convergence of solutions of such stochastic equations, if ε tends to zero to a some stochastic equations involving a local time of process, are obtained.

Keywords: Random process; stochastic equation; local time; convergence of measures.

AMS Subject Classification: 22E46, 53C35, 57S20

1. Introduction

It is well known (see [20]) that convergence of coefficients of stochastic differential equations is not sufficed for weak convergence of solutions of Ito's stochastic equation. It is necessary additional condition for diffusion coefficient of stochastic equation. For example, in [20, Theorem 11.3.3], authors supposed that diffusion coefficient satisfies the Lipschitz condition.

Kulinich [10] and Portenko [18] considered Ito's stochastic equation

$$x_\varepsilon(t) = x + \int_0^t b_\varepsilon(x_\varepsilon(s))ds + \int_0^t \sigma_\varepsilon(x_\varepsilon(s))dw(s).$$

In [10], the author established necessary and sufficient conditions for the weak convergence of certain functionals of solutions to these equations to one-dimensional Ito stochastic equations. Furthermore, the work addressed the weak convergence of the processes x_ε to generalized diffusion processes. These processes, as defined by Portenko [17], are continuous Markov processes whose transition probability densities are fundamental solutions to parabolic partial differential equations with drift coefficients given by generalized functions (specifically, distributions of the Dirac δ-function type).

In [18, Sec. 3, Chap. III], the weak convergence of the function $b_\varepsilon(x)$ to the Dirac δ-function concentrated at zero (with $\sigma_\varepsilon(x) \equiv 1$) was analyzed, and the weak convergence of the processes x_ε to generalized diffusion processes was proved. Finally, Makhno [12, 14] established the requirements for the convergence of measures generated by solutions of Ito stochastic equations to the measure generated by solutions of certain stochastic equations involving local time.

In this paper, we consider a stochastic equation involving local time. It was first investigated by Harrison and Shepp [3] and Le Gall [11]. In these works, solution of stochastic process with local time was connected with skew Brownian motion, defined by Ito and McKean [4]. Moreover in papers by Le Gall [11], Engelbert and Schmidt [1], Makhno [13] formulae which connect solutions of stochastic equations with local time with solutions of Ito's stochastic equations were obtained.

These problems and connected topics were discussed also in the works by Makhno [15; 16, Chaps. 3–5], Krykun and Makhno [6] and Krykun [7–9]. Book by Makhno [16] as well as a book by Jacod and Shiryaev [5] also contain more detailed review of results of other authors.

Building upon this foundation, in this paper we focus on the weak solution to the following stochastic differential equation involving a local time term at the origin:

$$u(t) = x + \beta L^u(t,0) + \int_0^t b(u(s))ds + \int_0^t \sigma(u(s))dw(s), \tag{1.1}$$

where β is a constant and $b(x), \sigma(x)$ are measurable functions. This explicit formulation provides the necessary context for understanding how nonregular dependencies in the pre-limit equations lead to such specialized processes.

To analyze the weak convergence toward the process (1.1), we consider a sequence of stochastic processes $u_\varepsilon(t)$ defined by the following pre-limit equations involving a parameter $\varepsilon > 0$:

$$u_\varepsilon(t) = x + \int_0^t (b_\varepsilon(u_\varepsilon(s)) + g_\varepsilon(u_\varepsilon(s)))ds + \int_0^t \sigma_\varepsilon(u_\varepsilon(s))dw(s), \tag{1.2}$$

where the functions b_ε are assumed to be unbounded (as $\varepsilon \to 0$) part of drift coefficients of the processes u_ε and the functions g_ε are assumed to be regular part of drift coefficients of the processes u_ε.

The behavior of these processes is primarily characterized by the associated scale function $f_\varepsilon(x)$, defined as

$$f_\varepsilon(x) = \int_0^x \exp\left\{-2\int_0^y \frac{b_\varepsilon(z)}{\sigma_\varepsilon^2(z)}dz\right\} dy.$$

The primary contribution of our study is the generalization of results by Portenko [17, 18], Kulinich [10] and Makhno[12, 14, 15]. The "nonregularity" investigated in this study pertains to the drift coefficients b_ε being unbounded with respect to the parameter ε as $\varepsilon \to 0$. Mathematically, this singularity is characterized by a lack of uniform Lipschitz continuity and the potential for the drift to concentrate into a distribution at the origin, which directly leads to the emergence of the local time term in the limit process (1.1) described above.

A primary innovation of our approach lies in the flexible treatment of the scale function f_ε. In contrast to previous studies where the limit function $f(x) = \lim_{\varepsilon\to 0} f_\varepsilon(x)$ is typically assumed to be linear, we permit $f(x)$ to converge to a nonlinear limit. This nonlinearity is a significant feature, as the derivatives of the limit function, $f_1 = f'(0-)$ and $f_2 = f'(0+)$, explicitly determine the intensity of the skewness constant β according to the following formula:

$$\beta = \frac{f_1 - f_2}{f_1 + f_2}.$$

This relationship demonstrates how the behavior of the scale functions at the point of singularity dictates the behavior of the resulting limit process.

Furthermore, by employing the general integral convergence requirements `(aa)` and `(aaa)` provided in Theorem 4.1, we establish a unified framework that handles variable and oscillatory coefficients, such as $\sigma^2(x) = 2 + \cos x$. To demonstrate the robustness of our results, a concrete example is provided in Sec. 5, which illustrates cases that fall outside the scope of traditional constant-diffusion theories.

The remainder of the paper is organized as follows. Section 2 provides formal definitions and notations. Section 3 contains auxiliary lemmas. Section 4 presents the proof of the main theorem.

2. Definition and Notations

We will use standard notation $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathsf{P})$ for probability space with flow of σ-algebras $\mathfrak{F}_t, t \in [0,T]$ and $(w(t), \mathfrak{F}_t)$ for the standard one-dimensional Wiener process with respect to the filtration $\mathfrak{F}_t$.

The indicator function of a set A is denoted by $I_A(x)$.

Let us consider one-dimensional stochastic equation involving a local time

$$\xi(t) = x + \beta L^{\xi}(t,0) + \int_0^t b(\xi(s))ds + \int_0^t \sigma(\xi(s))dw(s), \tag{2.1}$$

where β is a constant and $b(x), \sigma(x)$ are measurable functions.

Definition 2.1. According to [1, Definition 4.7(1)], Eq. (2.1) has *a weak solution*, if for the given functions $b(x)$ and $\sigma(x)$ and the constant β there is a probability space $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathsf{P})$ with flow of σ-algebras $\mathfrak{F}_t$, $t \geq 0$, continuous semimartingale $(\xi(t), \mathfrak{F}_t)$ and standard one-dimensional Wiener process $(w(t), \mathfrak{F}_t)$ such that local time $L^{\xi}(t,0)$ defined as

$$L^{\xi}(t,0) = \lim_{\delta\to 0} \frac{1}{2\delta} \int_0^t I_{(-\delta,\delta)}(\xi(s))\sigma^2(\xi(s))ds \tag{2.2}$$

exists almost surely and Eq. (2.1) is fulfilled almost surely.

Let $\mathbb{C}[0,T]$ be the space of continuous functions on interval $[0,T]$ equipped with the uniform topology. We denote by $C_t, t \in [0,T]$ the corresponding Borel σ-algebra.

Definition 2.2. Let P_{v_ε} be the probability measures on the measurable space $(\mathbb{C}[0,T], C_t)$ induced by the processes v_ε. We say that v_ε converges weakly if the sequence of measures P_{v_ε} converges weakly to the measure P_v as $\varepsilon \to 0$. Such kind of convergence we will denote as $\Rightarrow$ and will name *a weak convergence of random process.*

Definition 2.3. *The symmetric derivative of function* $h(x)$ is denoted as $\mathbb{D}h(x)$ and it is defined as

$$\mathbb{D}h(x) = \lim_{\varepsilon\to 0} \frac{h(x+\varepsilon) - h(x-\varepsilon)}{2\varepsilon}$$

provided that this limit exists and is finite for all x and *the second distributional derivative of function* $h(x)$ we denote as $\mathbf{n}_h(dx)$ and it is defined for every infinitely differentiable function $H(x)$ with compact support by the equality

$$\int \frac{d^2H(x)}{dx^2} h(x)dx = \int H(x)\mathbf{n}_h(dx).$$

Let us denote

$$\operatorname{sgn} x = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -1 & \text{for } x < 0. \end{cases}$$

If for some constants λ, Λ, $0 < \lambda \le \Lambda < \infty$ for the pair of measurable functions $(f(x), g(x))$, the following inequalities

$$|f(x)| \le \Lambda, \quad \lambda \le g(x) \le \Lambda$$

are fulfilled for every x, then we will denote this as

$$(f, g) \in \mathcal{L}(\lambda, \Lambda).$$

3. Auxiliary Results

Let us consider a random process $X(t)$ as a solution of the following Ito's stochastic equation:

$$X(t) = X(0) + \int_0^t \alpha(X(s))ds + \int_0^t \gamma(X(s))dw(s). \tag{3.1}$$

We suppose that there exist some constants λ, Λ $(0 < \lambda \le \Lambda < \infty)$ such as the pair of function $(\alpha, \gamma^2) \in \mathcal{L}(\lambda, \Lambda)$, then by [1, Theorem 4.35], Eq. (3.1) has a unique weak solution.

It follows from [1, Formula 4.3] that if for function $f(x)$, there exist derivatives $\mathbb{D}f(x)$ and $\mathbf{n}_f(dx)$, then Tanaka's formula (generalized Ito's formula)

$$f(X(t)) = f(X(0)) + \int_0^t \mathbb{D}f(X(s))dX(s) + \frac{1}{2}\int L^X(t, y)\mathbf{n}_f(dy) \tag{3.2}$$

is fulfilled.

Let functions $u_i(x)$, $i = 1, 2$ be some twice differentiable functions and let they have such properties:

$$u_1(0) = u_2(0) = 0, \quad u_1'(x) > 0, \quad u_2'(x) > 0.$$

We define function $u(x)$ as follows:

$$u(x) = \begin{cases} u_1(x) & \text{if } x \le 0, \\ u_2(x) & \text{if } x \ge 0. \end{cases} \tag{3.3}$$

Further, we denote $u_i'(0) = u_i$ and $v(x) = u^{-1}(x)$ as an inverse function of function $u(x)$. Then we obtain

$$\begin{aligned} \mathbb{D}u(x) &= \frac{1}{2}(u_2'(x) + u_1'(x)) + \frac{1}{2}(u_2'(x) - u_1'(x))\operatorname{sgn} x, \\ \mathbf{n}_u(dx) &= (u_2 - u_1)\delta_0(x)dx + \mathbb{A}_u(x)dx, \end{aligned}$$

where we define the function $\mathbb{A}_u(x)$ as follows:

$$\mathbb{A}_u(x) = \frac{1}{2}[(u_2''(x) + u_1''(x)) + (u_2''(x) - u_1''(x))\operatorname{sgn} x], \tag{3.4}$$

and $\delta_0(x)$ is Dirac δ-function concentrated at zero.

Lemma 3.1. *We assume that the equality* $Y(t) = u(X(t))$ *takes place. Then the following equalities*:

$$L^Y(t,0) = \frac{u_1+u_2}{2}L^X(t,0), \tag{3.5}$$

$$Y(t) = Y(0) + \frac{u_2-u_1}{u_2+u_1}L^Y(t,0) + \int_0^t \widehat{\alpha}(Y(s))ds + \int_0^t \widehat{\gamma}(Y(s))dw, \tag{3.6}$$

are fulfilled, where function $\mathbb{A}_u(x)$ *is defined by* (3.4), *and we denote*

$$\widehat{\alpha}(x) = \mathbb{D}u(v(x))\alpha(v(x)) + \frac{1}{2}\gamma^2(v(x))\mathbb{A}_u(v(x)),$$
$$\widehat{\gamma}(x) = \mathbb{D}u(v(x))\gamma(v(x)).$$

Proof. For the function $u(x)$ and the process $X(t)$ by Tanaka's formula (3.2), we obtain

$$Y(t) = Y(0) + \int_0^t \left[\mathbb{D}u(X(s))\alpha(X(s)) + \frac{1}{2}\gamma^2(X(s))\mathbb{A}_u(X(s))\right] ds$$
$$+ \int_0^t \mathbb{D}u(X(s))\gamma(X(s))dw + \frac{u_2-u_1}{2}L^X(t,0). \tag{3.7}$$

Since $X(s) = v(Y(s))$ then equality (3.6) follows from (3.7) and (3.5). Let us prove (3.5). For this purpose, we will apply formula (3.2) to process (3.1) and the function $\overline{u}(x) = |u(x)|$. We get

$$\mathbb{D}\overline{u}(x) = \frac{1}{2}(u_2'(x) - u_1'(x)) + \frac{1}{2}(u_2'(x) + u_1'(x))\operatorname{sgn} x,$$
$$\mathbf{n}_{\overline{u}}(dx) = (u_2+u_1)\delta_0(x)dx + \widehat{A}_u(x)dx,$$

where we denote the function $\widehat{A}_u(x)$ as follows:

$$\widehat{A}_u(x) = \frac{1}{2}[(u_2''(x) - u_1''(x)) + (u_2''(x) + u_1''(x))\operatorname{sgn} x].$$

Thus, we get

$$|Y(t)| = |Y(0)| + \int_0^t \left[\mathbb{D}\overline{u}(X(s))\alpha(X(s)) + \frac{1}{2}\gamma^2(X(s))\widehat{A}_u(X(s))\right] ds$$
$$+ \int_0^t \mathbb{D}\overline{u}(X(s))\gamma(X(s))dw + \frac{u_2+u_1}{2}L^X(t,0). \tag{3.8}$$

By formulae (3.2) and (3.8), we get

$$L^Y(t,0) = \frac{u_2+u_1}{2}L^X(t,0) + \int_0^t \left[\mathbb{D}\overline{u}(X(s))\alpha(X(s)) + \frac{1}{2}\gamma^2(X(s))\widehat{A}_u(X(s))\right] ds$$
$$+ \int_0^t \mathbb{D}\overline{u}(X(s))\gamma(X(s))dw - \int_0^t \operatorname{sgn}Y(s)dY(s).$$

The last integral on the right-hand side of this equation is derived from formula (3.7) and we use such property of processes that $\operatorname{sgn} X(s) = \operatorname{sgn} Y(s)$:

$$\begin{aligned} L^Y(t,0) = \frac{u_2+u_1}{2}L^X(t,0) &+ \int_0^t \left[\mathbb{D}\overline{u}(X(s))\alpha(X(s)) + \frac{1}{2}\gamma^2(X(s))\widehat{A}_u(X(s))\right] ds \\ &+ \int_0^t \mathbb{D}\overline{u}(X(s))\gamma(X(s))dw + \int_0^t \frac{u_2-u_1}{2}\operatorname{sgn} X(s)L^X(ds,0) \\ &- \int_0^t \operatorname{sgn} X(s)\left[\mathbb{D}u(X(s))\alpha(X(s)) + \frac{1}{2}\gamma^2(X(s))\mathbb{A}_u(X(s))\right] ds \\ &- \int_0^t \operatorname{sgn} X(s)\mathbb{D}u(X(s))\gamma(X(s))dw. \end{aligned} \tag{3.9}$$

Since the function $L^X(s,0)$ increases only at the points s such as $X(s) = 0$, then

$$\int_0^t \operatorname{sgn} X(s)L^X(ds,0) = 0.$$

Further, we will group terms of Eq. (3.9) and we will continue transformation of (3.9):

$$\begin{aligned} &\int_0^t \Bigg[\mathbb{D}\overline{u}(X(s))\alpha(X(s)) + \frac{1}{2}\gamma^2(X(s))\widehat{A}_u(X(s)) \\ &\quad - \operatorname{sgn} X(s)\left[\mathbb{D}u(X(s))\alpha(X(s)) + \frac{1}{2}\gamma^2(X(s))\mathbb{A}_u(X(s))\right]\Bigg] ds \\ &\quad + \int_0^t [\mathbb{D}\overline{u}(X(s))\gamma(X(s)) - \operatorname{sgn} X(s)\mathbb{D}u(X(s))\gamma(X(s))]dw \\ &= \int_0^t \Bigg[\alpha(X(s))\Bigg(\frac{u_2'(X(s)) - u_1'(X(s))}{2} + \frac{u_2'(X(s)) + u_1'(X(s))}{2}\operatorname{sgn} X(s) \\ &\quad - \left(\frac{u_2'(X(s)) + u_1'(X(s))}{2} + \frac{u_2'(X(s)) - u_1'(X(s))}{2}\operatorname{sgn} X(s)\right)\operatorname{sgn} X(s)\Bigg) \\ &\quad + \frac{1}{4}\gamma^2(X(s))\{((u_2''(X(s)) - u_1''(X(s))) + (u_2''(X(s)) + u_1''(X(s)))\operatorname{sgn} X(s)) \\ &\quad - ((u_2''(X(s)) + u_1''(X(s))) + (u_2''(X(s)) - u_1''(X(s)))\operatorname{sgn} X(s))\}\operatorname{sgn} X(s)\Bigg] ds \\ &\quad + \int_0^t \gamma(X(s))\Bigg[\frac{u_2'(X(s)) - u_1'(X(s))}{2} + \frac{u_2'(X(s)) + u_1'(X(s))}{2}\operatorname{sgn} X(s) \\ &\quad - \left(\frac{u_2'(X(s)) + u_1'(X(s))}{2} + \frac{u_2'(X(s)) - u_1'(X(s))}{2}\operatorname{sgn} X(s)\right)\operatorname{sgn} X(s)\Bigg] dw \end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \left(\alpha(X(s)) I_{\{0\}}(X(s)) \frac{u_2'(X(s)) - u_1'(X(s))}{2} \right. \\
&\quad \left. + \frac{1}{4}\gamma^2(X(s)) I_{\{0\}}(X(s))(u_2''(X(s)) - u_1''(X(s))) \right) ds \\
&\quad + \int_0^t I_{\{0\}}(X(s)) \frac{u_2'(X(s)) - u_1'(X(s))}{2} \gamma(X(s)) dw \\
&= \frac{1}{4} \int_0^t I_{\{0\}}(X(s)) \gamma^2(X(s))(u_2''(X(s)) - u_1''(X(s)) ds \\
&\quad + \int_0^t I_{\{0\}}(X(s)) \frac{u_2'(X(s)) - u_1'(X(s))}{2} dX(s) \\
&= \frac{1}{4}\gamma^2(0)(u_2''(0) - u_1''(0)) \int_0^t I_{\{0\}}(X(s)) ds + \frac{u_2 - u_1}{2} \int_0^t I_{\{0\}}(X(s)) dX(s) = 0,
\end{aligned}
$$

since it follows from [2, Lemma 5, p. 590] that

$$
\int_0^t I_{\{0\}}(X(s)) ds = 0
$$

and also since the following result:

$$
\begin{aligned}
\int_0^t I_{\{0\}}(X(s)) dX(s) &= \int_0^t I_{\{0\}}(X(s)) \frac{\alpha(X(s))}{\gamma^2(X(s))} \gamma^2(X(s)) ds \\
&= \int_{\mathbb{R}} I_{\{0\}}(y) \frac{\alpha(y)}{\gamma^2(y)} L^X(t, y) dy = 0
\end{aligned}
$$

takes place. Here, the first equality is a result from [19, p. 217] and the last one follows from [19, Chap. VI, Corollary 1.6]. □

Definition 3.1. For some constant $|\beta| < 1$, we define a function $\kappa(x)$ as follows:

$$
\kappa(x) = \begin{cases} (1-\beta)x & \text{if } x < 0, \\ (1+\beta)x & \text{if } x \geq 0, \end{cases} \tag{3.10}
$$

and a function

$$
\varphi(x) = \begin{cases} \dfrac{x}{1-\beta} & \text{if } x < 0, \\ \dfrac{x}{1+\beta} & \text{if } x \geq 0 \end{cases}
$$

is the inverse function of $\kappa(x)$. Furthermore, for every function f, we will denote a function $\widetilde{f}$ as follows:

$$
\widetilde{f}(x) = \frac{f(\kappa(x))}{1 + \beta \operatorname{sgn} x}. \tag{3.11}
$$

Further, we consider two following equations:

$$\xi(t) = x + \beta L^{\xi}(t,0) + \int_0^t g(\xi(s))ds + \int_0^t \sigma(\xi(s))dw(s), \tag{3.12}$$

$$\zeta(t) = \varphi(x) + \int_0^t \widetilde{g}(\zeta(s))ds + \int_0^t \widetilde{\sigma}(\zeta(s))dw(s). \tag{3.13}$$

For the measurable functions $g(x), \sigma(x)$ and the constant β, we introduce the following condition.

Condition (I)

(I$_1$) β is a constant such that $|\beta| < 1$.
(I$_2$) There exist some constants λ, Λ $(0 < \lambda \leq \Lambda < \infty)$ such that the pair of the function $(g, \sigma^2) \in \mathcal{L}(\lambda, \Lambda)$.

It is well known [1, Theorem 4.35; 21] that if Condition (I) holds, then Eqs. (3.12) and (3.13) have unique weak solutions. The next important lemma follows directly from Lemma 3.1.

Lemma 3.2. *We assume that the coefficients of process* (3.12) *satisfy Condition* (I). *Then the process* $\kappa(\zeta(t))$ *is a solution of Eq.* (3.12).

From Lemmas 3.1 and 3.2, it follows that for the function u defined by (3.3) the following lemma is fulfilled.

Lemma 3.3. *We assume that the coefficients of process* (3.12) *satisfy Condition* (I). *Then the process* $\eta(t) = u(\xi(t))$ *satisfies the following equation*:

$$\eta(t) = u(x) + \frac{u_2 - u_1 + \beta(u_2 + u_1)}{u_2 + u_1 + \beta(u_2 - u_1)} L^{\eta}(t,0) + \int_0^t g^*(\eta(s))ds + \int_0^t \sigma^*(\eta(s))dw(s), \tag{3.14}$$

where we denote

$$g^*(x) = \mathbb{D}u(u^{-1}(x))g(u^{-1}(x)) + \frac{\sigma^2(u^{-1}(x))}{2}\mathbb{A}_u(u^{-1}(x)),$$
$$\sigma^*(x) = \mathbb{D}u(u^{-1}(x))\sigma(u^{-1}(x)).$$

Proof. We observe that equalities

$$\eta(t) = u(\xi(t)) = u(\kappa(\zeta(t))) = \tau(\zeta(t))$$

take place, where function $\tau(x)$ is defined by equality

$$\tau(x) = u(\kappa(x)) = \begin{cases} u_1((1-\beta)x) & \text{if } x \leq 0, \\ u_2((1+\beta)x) & \text{if } x \geq 0. \end{cases}$$

Thus, we obtain the following results:

$$\begin{aligned}
&\tau_1(0) = \tau_2(0) = 0,\\
&\tau_1'(x) = (1-\beta)u_1'((1-\beta)x) > 0, \quad \tau_2'(x) > 0,\\
&\tau_1'(0) = \tau_1 = (1-\beta)u_1, \quad \tau_2'(0) = \tau_2 = (1+\beta)u_2,
\end{aligned}$$

there exists an inverse function of function $\tau(x)$ that is defined as $\tau^{-1}(x) = \varphi(u^{-1}(x))$; and there exist second derivatives $\tau_1''(x)$, $\tau_2''(x)$.

Further, we get

$$\begin{aligned}
\mathbb{D}\tau(x) &= (1+\beta\operatorname{sgn} x)\mathbb{D}u(\kappa(x)),\\
\mathbb{A}_\tau(x) &= (1+\beta\operatorname{sgn} x)^2\mathbb{A}_u(\kappa(x)),\\
\mathbf{n}_\tau(dx) &= (\tau_2-\tau_1)\delta_0(x)dx + (1+\beta\operatorname{sgn} x)^2\mathbb{A}_u(\kappa(x))dx.
\end{aligned}$$

In such a way, we conclude that the requirements of Lemma 3.1 on Eqs. (3.13) and (3.14) with the function $\tau(x)$ are fulfilled. Therefore, we obtain

$$\begin{aligned}
\widehat{\alpha}(x) &= \mathbb{D}\tau(\tau^{-1}(x))\widetilde{g}(\tau^{-1}(x)) + \frac{1}{2}\widetilde{\sigma}^2(\tau^{-1}(x))\mathbb{A}_\tau(\tau^{-1}(x))\\
&= (1+\beta\operatorname{sgn}\tau^{-1}(x))\mathbb{D}u(\kappa(\tau^{-1}(x)))\frac{g(\kappa(\tau^{-1}(x)))}{1+\beta\operatorname{sgn}\tau^{-1}(x)}\\
&\quad + \frac{\sigma^2(\kappa(\tau^{-1}(x)))}{2(1+\beta\operatorname{sgn}(\tau^{-1}(x)))^2}(1+\beta\operatorname{sgn}\tau^{-1}(x))^2\mathbb{A}_u(\kappa(\tau^{-1}(x)))\\
&= \mathbb{D}u(u^{-1}(x))g(u^{-1}(x)) + \frac{\sigma^2(u^{-1}(x))}{2}\mathbb{A}_u(u^{-1}(x)) = g^*(x),\\
\widehat{\gamma}(x) &= \mathbb{D}\tau(\tau^{-1}(x))\widetilde{\sigma}(\tau^{-1}(x))\\
&= (1+\beta\operatorname{sgn}\tau^{-1}(x))\mathbb{D}u(\kappa(\tau^{-1}(x)))\frac{\sigma(\kappa(\tau^{-1}(x)))}{1+\beta\operatorname{sgn}\tau^{-1}(x)}\\
&= \mathbb{D}u(u^{-1}(x))\sigma(u^{-1}(x)) = \sigma^*(x).
\end{aligned}$$

Finally, by applying Lemma 3.1, we get statement of Lemma 3.3. □

4. Main Result

Let us consider the following Ito's stochastic differential equation:

$$v_\varepsilon(t) = x + \int_0^t \big(b_\varepsilon(v_\varepsilon(s)) + g_\varepsilon(v_\varepsilon(s))\big)ds + \int_0^t \sigma_\varepsilon(v_\varepsilon(s))dw(s). \tag{4.1}$$

For measurable functions $g_\varepsilon(x), b_\varepsilon(x), \sigma_\varepsilon(x)$, we introduce the following condition.

Condition (II)

(II$_1$) For every $\varepsilon > 0$, there exists a unique weak solution of stochastic equation.

(II$_2$) There exist some constants λ, Λ $(0 < \lambda \leq \Lambda < \infty)$ such that the pair of the function $(g_\varepsilon, \sigma_\varepsilon^2) \in \mathcal{L}(\lambda, \Lambda)$.

(II$_3$) There exists a constant Λ_1 such that

$$\left| \int_0^x \frac{b_\varepsilon(y)}{\sigma_\varepsilon^2(y)} dy \right| \leq \Lambda_1$$

for every $x \in \mathbb{R}$ and $\varepsilon > 0$.

The assumptions on the singular drift b_ε are designed to characterize the behavior of the process at the limit. Specifically, Condition II$_3$ ensures that the integral of the ratio of the singular drift to the diffusion remains bounded, which prevents the process from exploding and enables the further definition of the scale function f_ε. To demonstrate the necessity of our generalized framework, we provide the example in the end of paper which falls outside the scope of classical Lipschitz-based or constant-diffusion theories.

We assume that coefficients of stochastic equation (4.1) satisfy Condition (II). By ν_ε we denote measure on functional space $(\mathbb{C}[0,T], \mathcal{C}_t)$ generated by process v_ε.

Let us define the scale functions $F_\varepsilon(x)$ and $f_\varepsilon(x)$ as follows:

$$F_\varepsilon(x) = \exp\left\{ -2 \int_0^x \frac{b_\varepsilon(y)}{\sigma_\varepsilon^2(y)} dy \right\}, \quad f_\varepsilon(x) = \int_0^x F_\varepsilon(y) dy. \tag{4.2}$$

Since $f_\varepsilon(x)$, as a function of x, is monotonically increasing, there exists an inverse function, which we will denote as $f_\varepsilon^{-1}(x)$.

Further, we assume that

$$\lim_{\varepsilon \to 0} f_\varepsilon(x) = f(x) = \begin{cases} f_1(x) & \text{if } x \leq 0, \\ f_2(x) & \text{if } x \geq 0, \end{cases} \tag{4.3}$$

and we assume also that functions $f_1(x), f_2(x)$ are twice differentiable functions such as $f_1(0) = f_2(0) = 0$; $f_1'(x) > 0$ (for $x \leq 0$) and $f_2'(x) > 0$ (for $x \geq 0$); and let us denote $f_1 = f_1'(0)$, $f_2 = f_2'(0)$.

Lemma 4.1. *Assume that Condition* II$_3$ *is satisfied. If the functions $f_\varepsilon(x)$ defined by* (4.2) *converge pointwise to a function $f(x)$ as $\varepsilon \to 0$, then this convergence is uniform on every compact set $K \subset \mathbb{R}$.*

Proof. Recall that $f_\varepsilon(x) = \int_0^x F_\varepsilon(y) dy$, where $F_\varepsilon(x) = \exp\{-2 \int_0^x \frac{b_\varepsilon(y)}{\sigma_\varepsilon^2(y)} dy\}$.

According to Condition II$_3$, there exists a constant Λ_1 such that $| \int_0^x \frac{b_\varepsilon(y)}{\sigma_\varepsilon^2(y)} dy| \leq \Lambda_1$ for all $x \in \mathbb{R}$ and $\varepsilon > 0$. This implies that the family of functions $F_\varepsilon(x)$ is uniformly bounded:

$$\exp\{-2\Lambda_1\} \leq F_\varepsilon(x) \leq \exp\{2\Lambda_1\}.$$

Consequently, for any x_1, x_2 in a compact set K, we have

$$|f_\varepsilon(x_1) - f_\varepsilon(x_2)| = \left| \int_{x_1}^{x_2} F_\varepsilon(y) dy \right| \le \exp\{2\Lambda_1\}|x_1 - x_2|.$$

This shows that the family $\{f_\varepsilon(x)\}$ is equicontinuous on K with a Lipschitz constant independent of ε. Since the family $\{f_\varepsilon(x)\}$ is also uniformly bounded ($|f_\varepsilon(x)| \le |x| \exp\{2\Lambda_1\}$), by the Arzelà–Ascoli theorem, the pointwise convergence $f_\varepsilon(x) \to f(x)$ implies uniform convergence on the compact set K.

Lemma 4.1 is proved. □

By Lemma 4.1, if the condition II_3 is fulfilled, then the limit in (4.3) exists uniformly on compact sets. Then

$$\lim_{\varepsilon \to 0} f_\varepsilon^{-1}(x) = f^{-1}(x)$$

uniformly on compact sets and $f^{-1}(x)$ is defined as inverse function for function $f(x)$ defined in (4.3).

Let β be a constant such that $|\beta| < 1$; $g(x)$ and $\sigma(x)$ be measurable functions such that the pair (g, σ^2) belongs to the class $L(\lambda, \Lambda)$ (for some constants λ, Λ, $0 < \lambda \le \Lambda < \infty$).

We will prove below that the limit measure ν for measures ν_ε on the space $(\mathbb{C}[0,T], \mathcal{C}_t)$ is generated by stochastic process involving local time defined by the following equation:

$$v(t) = x + \beta L^v(t, 0) + \int_0^t g(v(s)) ds + \int_0^t \sigma(v(s)) dw(s), \tag{4.4}$$

where the specific sense in which g and σ act as limits of g_ε and σ_ε is determined by requirements `(aa)` and `(aaa)` of Theorem 4.1.

Theorem 4.1. *We assume that the coefficients of process* (4.1) *satisfy Condition* (II), *the coefficients of process* (4.4) *satisfy Condition* (I), *and function $f(x)$ is defined in* (4.3). *For weak convergence of measures generated by processes $\nu_\varepsilon \Rightarrow \nu$ if $\varepsilon \to 0$, it is necessary and sufficient fulfilling of the following requirements*:

`(a)` $\beta = \dfrac{f_1 - f_2}{f_1 + f_2}$,

`(aa)` $\displaystyle\lim_{\varepsilon \to 0} \int_0^x \frac{1}{F_\varepsilon(y)\sigma_\varepsilon^2(y)} dy = \int_0^x \frac{1}{\sigma^2(y)\mathbb{D}f(y)} dy \quad for\ every\ x \in \mathbb{R}$,

`(aaa)` $\displaystyle\lim_{\varepsilon \to 0} \int_0^x \frac{g_\varepsilon(y)}{\sigma_\varepsilon^2(y)} dy = \int_0^x \left[\frac{g(y)}{\sigma^2(y)} + \frac{1}{2} \frac{\mathbb{A}_f(y)}{\mathbb{D}f(y)} \right] dy \quad for\ every\ x \in \mathbb{R}$.

Proof. Let us define the following process $\pi_\varepsilon(t) = f_\varepsilon(v_\varepsilon(t))$ and denote by ς_ε the measure generated by process π_ε on the space $(\mathbb{C}[0,T], C_t)$. It follows from Ito's formula that for the process $\pi_\varepsilon(t)$, the following equation:

$$\pi_\varepsilon(t) = \pi_\varepsilon(0) + \int_0^t \widehat{g_\varepsilon}(\pi_\varepsilon(s))ds + \int_0^t \widehat{\sigma_\varepsilon}(\pi_\varepsilon(s))dw(s) \tag{4.5}$$

is fulfilled. Here we denote

$$\widehat{g_\varepsilon}(x) = F_\varepsilon(f_\varepsilon^{-1}(x))g_\varepsilon(f_\varepsilon^{-1}(x)),$$
$$\widehat{\sigma_\varepsilon}(x) = F_\varepsilon(f_\varepsilon^{-1}(x))\sigma_\varepsilon(f_\varepsilon^{-1}(x)).$$

Let us define $\pi(t) = f(v(t))$ and denote by ς the measure generated by process π on the functional space $(\mathbb{C}[0,T], C_t)$.

Further, we will use [14, Lemma 5] so we need to verify that requirements `(a)`, `(aa)`, `(aaa)` are necessary and sufficient conditions for convergence of measures ς_ε to the measure ς. Due to the fact that Condition (I) is fulfilled then the pair of functions $(\widehat{g_\varepsilon}, \widehat{\sigma_\varepsilon}^2) \in \mathcal{L}(a, A)$ for some constants a, A.

It follows from Lemma 3.3 that process $\pi(t) = f(v(t))$ satisfies Eq. (3.14) with such expression for the constant at the local time (one should substitute constants u_1, u_2, β by constants f_1, f_2, β, respectively):

$$\pi(t) = \pi(0) + \frac{f_2 - f_1 + \beta(f_2 + f_1)}{f_2 + f_1 + \beta(f_2 - f_1)} L^\pi(t, 0) + \int_0^t \widehat{g}(\pi(s))ds + \int_0^t \widehat{\sigma}(\pi(s))dw(s),$$

where we denote

$$\widehat{g}(x) = \mathbb{D}f(f^{-1}(x))g(f^{-1}(x)) + \frac{\sigma^2(f^{-1}(x))}{2}\mathbb{A}_f(f^{-1}(x)),$$
$$\widehat{\sigma}(x) = \mathbb{D}f(f^{-1}(x))\sigma(f^{-1}(x)).$$

From another hand, it follows from the results by Makhno [12] that the limit process for $\pi_\varepsilon(t)$ is a solution of Ito's stochastic equations, so constant at the local time of limit process has to be equal to zero. In such a way, we can calculate a constant β. Thus, we conclude that requirement `(a)` of the theorem has to be fulfilled.

Further from [12, Theorem 1], we conclude that the following requirements are necessary and sufficient conditions for convergence of measures $\varsigma_\varepsilon \Rightarrow \varsigma$ if $\varepsilon \to 0$:

$$\begin{aligned}\int_0^x \frac{1}{\widehat{\sigma_\varepsilon}^2(y)}dy &= \int_0^x \frac{1}{F_\varepsilon^2(f_\varepsilon^{-1}(y))\sigma_\varepsilon^2(f_\varepsilon^{-1}(y))}dy \\ &= \int_0^{f_\varepsilon^{-1}(x)} \frac{1}{F_\varepsilon(z)\sigma_\varepsilon^2(z)}dz \to \int_0^{f^{-1}(x)} \frac{1}{\mathbb{D}f(z)\sigma^2(z)}dz \\ &= \int_0^x \frac{1}{\sigma^2(f^{-1}(y))\left[\mathbb{D}f(f^{-1}(y))\right]^2}dy = \int_0^x \frac{1}{\widehat{\sigma}^2(y)}dy,\end{aligned}$$

$$\begin{aligned}
\int_0^x \frac{\widehat{g_\varepsilon}(y)}{\widehat{\sigma_\varepsilon}^2(y)}dy &= \int_0^x \frac{g_\varepsilon(f_\varepsilon^{-1}(y))}{F_\varepsilon(f_\varepsilon^{-1}(y))\sigma_\varepsilon^2(f_\varepsilon^{-1}(y))}dy \\
&= \int_0^{f_\varepsilon^{-1}(x)} \frac{g_\varepsilon(z)}{\sigma_\varepsilon^2(z)}dz \to \int_0^{f^{-1}(x)} \left[\frac{g(z)}{\sigma^2(z)} + \frac{1}{2}\frac{\mathbb{A}_f(z)}{\mathbb{D}f(z)}\right] dz \\
&= \int_0^x \left[\frac{g(f^{-1}(y))}{\sigma^2(f^{-1}(y))} + \frac{1}{2}\frac{\mathbb{A}_f(f^{-1}(y))}{\mathbb{D}f(f^{-1}(y))}\right]\frac{dy}{\mathbb{D}f(f^{-1}(y))} \\
&= \int_0^x \left[\frac{g(f^{-1}(y))}{\sigma^2(f^{-1}(y))\mathbb{D}f(f^{-1}(y))} + \frac{1}{2}\frac{\mathbb{A}_f(f^{-1}(y))}{\left[\mathbb{D}f(f^{-1}(y))\right]^2}\right] dy \\
&= \int_0^x \frac{\widehat{g}(y)}{\widehat{\sigma}^2(y)}dy,
\end{aligned}$$

that is equivalent to requirements `(aa)`, `(aaa)` of the theorem. □

5. Example

The purpose of this section is to demonstrate the necessity of our generalized framework by providing an example that falls outside the scope of classical Lipschitz-based or constant-diffusion theories.

Let $v_\varepsilon(t)$ be the solution of the following stochastic differential equation:

$$v_\varepsilon(t) = x + \int_0^t b_\varepsilon(v_\varepsilon(s))ds + \int_0^t \sigma(v_\varepsilon(s))dw(s), \tag{5.1}$$

where the coefficients are defined as

$$\begin{aligned}
b_\varepsilon(x) &= \frac{K}{\varepsilon} I_{[0;\varepsilon]}(x), \quad \text{with a constant } K \neq 0, \\
\sigma(x) &= \sqrt{2 + \cos x}.
\end{aligned}$$

In this example, $g_\varepsilon(x) \equiv 0$ so the pair of functions $(g_\varepsilon, \sigma^2)$ belongs to the class $L(\lambda, \Lambda)$ with $\lambda = 1$ and $\Lambda = 3$. So, Condition II_2 is fulfilled.

Further, let us verify the uniform boundedness of the integral in Condition II_3:

$$\left|\int_0^x \frac{b_\varepsilon(y)}{\sigma^2(y)}dy\right| = \left|\int_0^x \frac{\frac{K}{\varepsilon}I_{[0;\varepsilon]}(y)}{2 + \cos y}dy\right|.$$

For $x > \varepsilon$, the integral equals $|\frac{K}{\varepsilon}\int_0^\varepsilon \frac{1}{2+\cos y}dy|$. As $\varepsilon \to 0$, by the mean value theorem, this expression tends to $\frac{K}{2+\cos 0} = \frac{K}{3}$. Thus, the integral is uniformly bounded for all $\varepsilon > 0$, satisfying Condition II_3.

According to (4.2) and (4.3), we calculate the auxiliary functions $F_\varepsilon(x)$, $f_\varepsilon(x)$ and $f(x)$. By using the results above, $\varepsilon \to 0$.

$$\text{For } x < 0, \quad F_\varepsilon(x) = 1, \quad \text{hence } f(x) = x \text{ and } f_1 = f'(0-) = 1,$$

$$\text{For } x > 0, \quad F_\varepsilon(x) \to \exp\left\{-\frac{2K}{3}\right\}, \quad \text{so } f(x) = x\exp\left\{-\frac{2K}{3}\right\},$$

$$f_2 = f'(0+) = \exp\left\{-\frac{2K}{3}\right\}.$$

By applying Theorem 4.1, the measures ν_ε, generated by process v_ε on functional space $(\mathbb{C}[0,T],\mathcal{C}_t)$, converge weakly to the measure ν generated by the process:

$$v(t) = x + \beta L^v(t,0) + \int_0^t \sigma(v)dw(s),$$

where local time constant we calculate following condition `(a)` of Theorem 4.1:

$$\beta = \frac{f_1 - f_2}{f_1 + f_2} = \frac{1-\exp\{-\frac{2K}{3}\}}{1+\exp\{-\frac{2K}{3}\}} = \tanh\left(\frac{K}{3}\right).$$

Further, since $\sigma_\varepsilon(x) = \sigma(x)$, requirement `(aa)` of Theorem 4.1 is trivially satisfied with $\sigma(x) = \sqrt{2+\cos x}$.

Finally, requirement `(aaa)` of Theorem 4.1 is satisfied with $g(x) = 0$.

This example demonstrates that the intensity of the local time β is determined by the specific value of the diffusion coefficient at the point of singularity ($\sigma^2(0) = 3$). If the diffusion were $\sigma^2(0) = 1$, the local time constant would be $\tanh(K)$; however, the presence of variable diffusion dampens or amplifies the “push” of the singular drift, a generalization that distinguishes this work from standard results in the literature [10, 12, 14, 17, 18].

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