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Convergence of skew Brownian motions with local times at several points that are contracted into a single one

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Presented by S. Ya. Makhno

Abstract. Conditions of convergence in mean of skew Brownian motions with local times at several points that are contracted into a limit point are obtained. It is proved that the limit process is also a skew Brownian motion with local time at the limit point. A formula to calculate the coefficient of the local time of the limit process is given.

Keywords. Stochastic equation, local time, skew Brownian motion, convergence in mean.

1. Introduction

Consider the skew Brownian motion as a solution of the stochastic equation with local times at N points and with coefficients depending on the parameter n

$$\xi_n(t) = \beta_1(n) L^{\xi_n}(t, 0) + \sum_{i=2}^{N} \beta_i(n) L^{\xi_n}(t, a_i(n)) + w(t), \quad t \in [0, T]. \tag{1.1}$$

We will study the question about the convergence of solutions of the stochastic equation (1.1) under the condition that, as the parameter $n \to \infty$, the coefficients of the local times $\beta_i(n)$ tend, in this case, to their limit values β_i $(i = 1, ..., N)$, respectively, and the points $a_i(n)$ tend to 0 $(i = 2, ..., N)$.

Here, we will consider the skew Brownian motion defined by K. Itô and H. McKean [1] and constructed in connection with Feller's classification of one-dimensional diffusion processes in terms of elliptic second-order differential operators. In the works by W. A. Rosenkrantz [2] and M. I. Portenko [3], the skew Brownian motion was obtained as a weak limit of a certain process whose drift coefficient tends to the delta-function at the point 0. Later on, the skew Brownian motion was considered in a lot of works. We mention the works by J. M. Harrison and L. A. Shepp [4] and J.-F. Le Gall [5], where this process was related to the solution of a stochastic equation with local time. In work [5] and in works by H.-J. Engelbert and W. Schmidt [6] and S. Ya. Makhno [7], the formulas presenting the relation of solutions of the stochastic equations with local time to solutions of the Itô equations are proposed. The detailed review of recent results, numerous generalizations, and properties of a skew Brownian motion can be found in works by A. Lejay [8] and J. M. Ramirez [9].

M. I. Portenko [10, 11] considered the solutions of stochastic equations of the type (1.2) as diffusion processes in a space with semipermeable transparent barriers (semipermeable membranes).

The question about the behavior of process (1.1) in a partial case, namely, a skew Brownian motion with two semitransparent membranes contracting into a single one, i.e., a process of the form

$$\xi_n(t) = \beta_1(n) L^{\xi_n}(t, 0) + \beta_2(n) L^{\xi_n}(t, a(n)) + w(t), \ \ t \in [0, T], \tag{1.2}$$

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where $a(n) \to 0$ as $n \to \infty$, was considered by L. L. Zaitseva [12]. Her result coincides with the result in an analogous case obtained in this work (see Example 1). D. Dereudre, S. Mazzonetto, and S. Roelly in [13] considered and calculated the transient density of a skew Brownian motion with two semipermeable barriers (and with constant drift).

In the recent works [14] and [15], S. Ya. Makhno considered the question about the limiting behavior of the solutions of stochastic equations with local times at many points (with barriers) in the cases where the barriers fill a certain segment [14] in the limit case or converge into a single one [15]. The equations for the processes in [14,15] are of the general form, but only the weak convergence to a limiting process was established with their use. Here, we will prove the convergence of a skew Brownian process to the limit process in mean.

It is known [4] that, for $|\beta_i| > 1$, Eq. (1.1) has no solutions. But if $|\beta_i| = 1$, this situation corresponds, according to [16], to the presence of impermeable membrane at the point $a_i(n)$. Therefore, the question about the limit process for process (1.1) in the case where $|\beta_i| < 1$, but $\left|\sum_{i=1}^{N} \beta_i\right| \geq 1$, is of special interest. Our result shows that the limit process will be also a skew Brownian motion with the coefficient of a local time whose modulus is less than 1. Moreover, the formula for the mentioned coefficient of a local time is given.

The work is organized as follows: in Section 2, we give the notations and formulate the main result, Theorem 1; in Section 3, Theorem 1 and the auxiliary lemmas 1—2 are proved. Section 4 contains the conclusions and generalizations of the results. In Section 5, we present some model examples.

2. Main result

We now introduce the necessary notations. Let $I_A(x)$ be an indicator of the set A.

As fixed n and $|\beta_i(n)| \leq 1$, $i = 1, ..., N$, Eq. (1.1) has the unique strong solution [4], [17, Theorem II.5.5]. In other words, on the probabilistic space $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathsf{P})$ with a flow of σ-algebras $\mathfrak{F}_t, t \in [0, T]$ and the given standard one-dimensional Wiener process $(w(t), \mathfrak{F}_t)$, there exists a continuous semimartingal $(\xi(t), \mathfrak{F}_t)$ such that the symmetric local times at the points 0 and $a_i(n)$, $i = 2, ..., N$ that are set by the equality

$$L^{\xi_n}(t, b) = \lim_{\delta \to 0} \frac{1}{2\delta} \int_0^t I_{\left(b-\delta, b+\delta\right)}(\xi_n(s))ds,$$

exist almost surely, and (1.1) holds almost surely.

For the skew Brownian motion (1.1), we introduce the following condition.

Condition (I):

I_1. $|\beta_i(n)| < 1$ for all n and $i = 1, ..., N$.

I_2. There exist constants β_i such that $|\beta_i| < 1$, $i = 1, ..., N$, and

$$\lim_{n \to \infty} \beta_i(n) = \beta_i, \quad i = 1, ..., N.$$

I_3. $a_i(n) > 0$ for all n, $\quad i = 2, ..., N$.

I_4. $a_i(n) \neq a_j(n)$ for $i \neq j \quad$ for all n, $\quad i, j = 2, ..., N$.

I_5. For $i = 2, ..., N$,

$$\lim_{n \to \infty} a_i(n) = 0$$

holds.

The main result of the present work is the following proposition.

Theorem 1. *Let Condition (I) be satisfied for Eq.* (1.1). *Then the convergence of the processes of the skew Brownian motion* (1.1) *to the limit process*

$$\xi(t) = \gamma L^{\xi}(t,0) + w(t), \quad t \in [0,T], \tag{2.3}$$

holds in mean uniformly in time, as $n \to \infty$. In other words, the equality

$$\lim_{n\to\infty} \mathsf{E}\Big[\sup_{0\le t\le T} \big|\xi_n(s) - \xi(s)\big|\Big] = 0 \tag{2.4}$$

holds. In this case, the coefficient γ of the local time of the limit process $\xi(t)$ is given by the formula

$$\gamma = \frac{\prod_{i=1}^{N}(1+\beta_i) - \prod_{i=1}^{N}(1-\beta_i)}{\prod_{i=1}^{N}(1+\beta_i) + \prod_{i=1}^{N}(1-\beta_i)}. \tag{2.5}$$

We denote a hyperbolic tangent of x by $\tanh x$ and an areatangent (inverse hyperbolic tangent) of x by artanh x.

Remark 1. The coefficient γ of the local time of a limit process can be found by the formula

$$\gamma = \tanh\Big(\sum_{i=1}^{N} \operatorname{artanh} \beta_i\Big). \tag{2.6}$$

The identity of formulas (2.5) and (2.6) will be proved in Lemma 2 below.

Remark 2. It is simple to prove that if Condition I_2 is satisfied, the coefficient of the local time of a limit process is bounded in the same limits: $|\gamma| < 1$.

3. Proof of the main result

Proof of Theorem 1. The proof consists in the application of the result in [5, Theorem 3.1] to process (1.1).

Indeed, in notations from [5], we have

$$\varphi_n(t) \equiv 1, \quad \nu_n(dx) = \beta_1(n)\delta_0(x)dx + \sum_{i=2}^{N} \beta_i(n)\delta_{a_i(n)}(x)dx.$$

Thus, if Condition (I) is satisfied, φ_n and ν_n belong to the corresponding classes of functions/measures and are bounded according to the requirements in [5, Theorem 3.1].

Then, in notations from [5], we have

$$f_{\nu_n}(x) = \prod_{y\le x} \frac{1-\nu_n(\{y\})}{1+\nu_n(\{y\})} = \begin{cases} 1, & x < 0, \\ \dfrac{1-\beta_1(n)}{1+\beta_1(n)}, & 0 \le x < a_2(n), \\ \prod_{i=1}^{k-1} \dfrac{1-\beta_i(n)}{1+\beta_i(n)}, & a_{k-1}(n) \le x < a_k(n), \qquad k = 3,4,\dots,N, \\ \prod_{i=1}^{N} \dfrac{1-\beta_i(n)}{1+\beta_i(n)}, & x \ge a_N(n). \end{cases}$$

The limit functions are

$$\varphi(t) \equiv 1, \quad f(x) = \begin{cases} 1, & x < 0 \\ \prod_{i=1}^{N} \dfrac{1-\beta_i}{1+\beta_i}, & x \geq 0. \end{cases}$$

From whence, the bounded measure associated with this function f is

$$f'(dx) = \Big(f(0) - f(0-)\Big)\delta_0(x)dx = \Big(\prod_{i=1}^{N} \frac{1-\beta_i}{1+\beta_i} - 1\Big)\delta_0(x)dx.$$

Finally, we define the measure $\nu(dx)$ as follows:

$$\nu(dx) = -\frac{f'(dx)}{f(0)+f(0-)} = \frac{1 - \prod\limits_{i=1}^{N} \dfrac{1-\beta_i}{1+\beta_i}}{1 + \prod\limits_{i=1}^{N} \dfrac{1-\beta_i}{1+\beta_i}} \delta_0(x)dx.$$

Thus, all requirements in [5, Theorem 3.1] are satisfied, which yields the validity of equality (2.4), where the limit process $\xi(t)$ has the form (2.3) with the coefficient of the local time given by the formula

$$\gamma = \frac{1 - \prod\limits_{i=1}^{N} \dfrac{1-\beta_i}{1+\beta_i}}{1 + \prod\limits_{i=1}^{N} \dfrac{1-\beta_i}{1+\beta_i}}.$$

From whence, it is easy to get formula (2.5) for the coefficient of the local time of a limit process. □

Lemma 1. *The coefficient γ of the local time of a limit process defined by formula* (2.5) *or* (2.6) *can be determined by the recurrence formula $\gamma = \gamma_N$, where*

$$\gamma_1 = \beta_1, \quad \gamma_i = \frac{\gamma_{i-1} + \beta_i}{1 + \gamma_{i-1}\beta_i}, \qquad i = 2, ..., N. \tag{3.7}$$

Proof. **1.** First, we prove the lemma for γ given by formula (2.5). To simplify the record, we denote

$$B_k^+ = \prod_{i=1}^{k} (1+\beta_i), \qquad B_k^- = \prod_{i=1}^{k} (1-\beta_i).$$

Then formula (2.5) takes the form

$$\gamma = \gamma_N = \frac{B_N^+ - B_N^-}{B_N^+ + B_N^-}.$$

We now prove the lemma by induction.

For $k = 1$, the assertion of the lemma is obvious, $\gamma_1 = \beta_1$. Take $k = 2$. Hence, by formula (2.5),

$$\gamma_2 = \frac{B_2^+ - B_2^-}{B_2^+ + B_2^-} = \frac{(1+\beta_1)(1+\beta_2) - (1-\beta_1)(1-\beta_2)}{(1+\beta_1)(1+\beta_2) - (1-\beta_1)(1-\beta_2)} = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2};$$

and, by formula (3.7), we have

$$\gamma_2 = \frac{\gamma_1 + \beta_2}{1 + \gamma_1\beta_2} = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}.$$

We have got the identical values. Hence, the lemma is satisfied for $k = 2$.

Assume that the lemma holds for any integer $k - 1$:

$$\gamma_{k-1} = \frac{B^+_{k-1} - B^-_{k-1}}{B^+_{k-1} + B^-_{k-1}}.$$

Formulas (2.5) and (3.7) yield

$$\gamma_k = \frac{\gamma_{k-1} + \beta_k}{1 + \gamma_{k-1}\beta_k} = \frac{\frac{B^+_{k-1}-B^-_{k-1}}{B^+_{k-1}+B^-_{k-1}} + \beta_k}{1 + \frac{B^+_{k-1}-B^-_{k-1}}{B^+_{k-1}+B^-_{k-1}}\beta_k} = \frac{B^+_{k-1} - B^-_{k-1} + \beta_k\left(B^+_{k-1} + B^-_{k-1}\right)}{B^+_{k-1} + B^-_{k-1} + \beta_k\left(B^+_{k-1} - B^-_{k-1}\right)}$$

$$= \frac{B^+_{k-1}\left(1 + \beta_k\right) - B^-_{k-1}\left(1 - \beta_k\right)}{B^+_{k-1}\left(1 + \beta_k\right) + B^-_{k-1}\left(1 - \beta_k\right)} = \frac{B^+_k - B^-_k}{B^+_k + B^-_k}.$$

Thus, we obtain that the assertion of the lemma is valid for k. By virtue of the arbitrariness of k, this completes the proof of the lemma for formula (2.5).

2. We now prove the lemma for γ given by formula (2.6). For a hyperbolic tangent, we have the formula

$$\tanh(A + B) = \frac{\tanh A + \tanh B}{1 + \tanh A \tanh B}. \tag{3.8}$$

By transforming the right-hand side, we get

$$\frac{\tanh A + \tanh B}{1 + \tanh A \tanh B} = \frac{\frac{e^A - e^{-A}}{e^A + e^{-A}} + \frac{e^B - e^{-B}}{e^B + e^{-B}}}{1 + \frac{e^A - e^{-A}}{e^A + e^{-A}} \cdot \frac{e^B - e^{-B}}{e^B + e^{-B}}} = \frac{(e^A - e^{-A})(e^B + e^{-B}) + (e^B - e^{-B})(e^A + e^{-A})}{(e^A + e^{-A})(e^B + e^{-B}) + (e^A - e^{-A})(e^B - e^{-B})}$$

$$= \frac{2\left(e^{A+B} - e^{-A-B}\right)}{2\left(e^{A+B} + e^{-A-B}\right)} = \tanh(A + B).$$

Then we proceed by induction. For $k = 1$, we have

$$\gamma_1 = \tanh\left(\operatorname{artanh} \beta_1\right) = \beta_1$$

by formula (2.6). In other words, the lemma holds for $k = 1$; for $k = 2$, we have

$$\gamma_2 = \tanh\left(\operatorname{artanh} \beta_1 + \operatorname{artanh} \beta_2\right) = \frac{\tanh\left(\operatorname{artanh} \beta_1\right) + \tanh\left(\operatorname{artanh} \beta_2\right)}{1 + \tanh\left(\operatorname{artanh} \beta_1\right) \tanh\left(\operatorname{artanh} \beta_2\right)} = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}$$

by formulas (2.6) and (3.8). Hence, the lemma is satisfied for $k = 2$. Finally, if the lemma is satisfied for any $k - 1$, $k \geq 3$, then formula (3.8) yields (3.7) for any k. By virtue of the arbitrariness of k, it completes the proof of the lemma for formula (2.6). □

Lemma 2. *The formulas for the coefficient of the local time of a limit process* (2.5) *and* (2.6) *are equivalent.*

Proof. The assertion of the lemma follows from Lemma 1 and the equality of γ_1 and γ_2 given by formulas (2.5) and (2.6). □

4. Conclusions and generalizations

The analysis of the proof of Theorem 1 gives possibility to draw the following conclusions.

Corollary 1. *One can omit condition* I_3 *(i.e.,* $a_i(n) > 0$*,* $i = 2, ..., N$*) retaining other items of Condition (I). Then the assertion of Theorem 1 remains valid, the same as formula* (2.5) *for the coefficient of the local time of a limit process.*

Corollary 2. *The limit point for the sequence* $a_i(n)$ *can be any point* α*, but not only the point* 0*. In other words, Condition* I_5 *can also be weakened. The assertion of Theorem 1 remains valid, the same as formula* (2.5) *for the coefficient of the local time of a limit process.*

Corollary 3. *Theorem 1 holds also in the case of the diffusion coefficient different from 1 for the process*

$$\xi_n(t) = x_0 + \sum_{i=1}^{N} \beta_i(n) L^{\xi_n}(t, a_i(n)) + \int_0^t \sigma_n(\xi_n(s)) dw(s), \ t \in [0, T], \tag{4.9}$$

for which the symmetric local times at the points $a_i(n)$*,* $i = 1, ..., N$*, are set by the equality*

$$L^{\xi_n}(t, b) = \lim_{\delta \to 0} \frac{1}{2\delta} \int_0^t I_{\left(b-\delta, b+\delta\right)}(\xi_n(s)) \sigma^2(\xi_n(s)) ds.$$

For process (4.9) *to converge to a limit process (and for the existence of a strong solution of Eq.* (4.9)*), it is necessary, in addition to the validity of Condition (I), that the diffusion coefficient satisfy the requirements in [5, Theorem 3.1]. In other words,* $\sigma_n(x)$ *should be:*
— a function of bounded variation;
— right continuous;
— bounded and separated from zero; i.e., a constant Λ *such that*

$$\frac{1}{\Lambda} \leq \sigma_n(x) \leq \Lambda$$

should exist;
— a function $\sigma(x)$ *such that*

$$\lim_{n \to \infty} \int_{-K}^{K} |\sigma_n(x) - \sigma(x)| dx = 0 \text{ for all } K > 0$$

should exist.

The limit process will be

$$\xi(t) = x_0 + \gamma L^{\xi}(t, 0) + \int_0^t \sigma(\xi(s)) dw(s), \ t \in [0, T], \tag{4.10}$$

with the coefficient γ *given by formula* (2.5) *or* (2.6).

5. Examples

Example 1. Consider the skew Brownian motion with two semitransparent membranes that are contracted into a single one. In other words, we consider process (1.2). Let it satisfy condition (I). Which is the limit process?

According to Theorem 1, the limit process for (1.2) is

$$\xi(t) = \gamma L^{\xi}(t,0) + w(t), \quad t \in [0,T],$$

where the coefficient of the local time is, by formula (2.5),

$$\gamma = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2}. \tag{5.11}$$

Example 2. Let us consider the problem: Which should the limits of the coefficients β_i of the local times of a skew Brownian motion (1.1) be in order that the coefficient of the local time of the limit skew Brownian motion (2.3) is equal to the given A, where $|A| < 1$?

It is clear that this problem can have more than one solution, because the coefficients of the local times of a sublimit process and their quantity can be arbitrary.

Let us have the skew Brownian motion with N local times (1.1) and with equal coefficients of the local times $\beta_i = \beta$, where $|\beta| < 1$, for $i = 1, ..., N$. Then it is more convenient to use formula (2.6):

$$A = \tanh(N \operatorname{artanh} \beta);$$

$$N \operatorname{artanh} \beta = \operatorname{artanh} A;$$

$$\operatorname{artanh} \beta = \frac{\operatorname{artanh} A}{N}.$$

From whence, we can get the final formula for the limit value of the coefficients of the local times:

$$\beta = \tanh\left(\frac{\operatorname{artanh} A}{N}\right).$$

Consider separately the simplest variant, namely, a skew Brownian motion with two local times (1.2) and with identical coefficients of the local times: $\beta_1 = \beta_2 = \beta$, where $|\beta| < 1$. Then, in order to find the coefficient β, it is more convenient to apply formula (5.11):

$$\frac{2\beta}{1+\beta^2} = A.$$

The simple analysis of the obtained equation quadratic in β shows that the limit value of the coefficients of the local times of a sublimit skew Brownian motion is as follows:

$$\beta = \frac{1 - \sqrt{1 - A^2}}{A}.$$

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